THE MATHEMATICAL WORK

OF

Dr. FERDINAND HURTER

BY

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AND

A Letter of 22nd August, 1897, from Dr. Hurter to Mr. Driffield dealing with the Theory of Development, annexed at the request of Dr. Allen.

Dr. HURTER'S MATHEMATICAL WORK.

The note books of Dr. Hurter now in the possession of the Royal Photographic Society contain many pages of mathematical analysis interspersed with records of experimental work. The most important part of the mathematical investigation is concerned with the fundamental formula given in the *J.S.C.I.*, 31st May, 1890. This formula expresses the density after development as a function of the opacity of the unexposed plate, of the time of exposure, and of the "inertia" of the silver bromide.

The physical assumptions on which the formula is based may be, and have been, made the subject of criticism.[1] Some of these assumptions, as that the film obeys the exponential law for the absorption of light, must be regarded merely as approximations. But given the assumptions, there can be no doubt as to the correctness of the mathematical development given by Dr. Hurter. In the following pages, which should be read in conjunction with the published papers, the mathematical reasoning given on various pages of the note books is abstracted. In some cases the notation differs slightly from that finally employed in the joint paper.

[1] *See* Sheppard and Mees, *Investigations on the Theory of the Photographic Process*, pp. 210, 288 (Longmans, 1907).

THE FUNDAMENTAL LAW.

Dr. Hurter's Note Book, H.N.—B., pp. 58 and 59. My law—[1]

$$dx = (1 - a)\frac{I}{i}[e^{-kx} - e^{-ka}]\, dt.$$

$(1 - a)$ = fraction of light not reflected.
I = intensity of light.
i = inertia of silver salt.
e^{-kx} = light not absorbed by changed silver.
e^{-ka} = light not absorbed by total silver.
$e^{-kx} - e^{-ka}$ is light passing by changed less that passing by all.
dt = differential of time.

Result of integration—

$$\log.\ \frac{e^{ka} - 1}{e^{ka} - e^{kx}} = \frac{k}{e^{ka}} \cdot (1 - a)\frac{It}{i}$$

Recapitulation of the law (p. 125).

Complete derivation of the law of action of light (p. 154).

$$\int_0^x \frac{dx}{e^{-kx} - e^{-ka}} = \int_0^t (1 - a)\frac{I\, dt}{i}$$

Put $e^{-kx} = y$ and $e^{-ka} = A$ then

$$dy = -e^{-kx}\, dx \,.\, k \text{ or}$$

$$dx = -\frac{dy}{y}\,\frac{1}{k} \text{ or}$$

$$\int_0^x \frac{dx}{c^{-kx} - e^{-ka}} = -\frac{1}{k}\int_1^{e^{-kx}} \frac{dy}{y\,(y - A)}$$

(3)

$$\int_1^{e^{-kx}} \frac{dy}{y(y - A)} = -k(1 - a)\frac{I}{i}t.$$

$$\int \frac{dy}{y\,(y - A)} = \frac{1}{A}\int \frac{dy}{y - a} - \frac{dy}{y}$$

$$\frac{1}{y\,(y - A)}. = \frac{c}{y} + \frac{B}{y - a}$$

$$1 = C\,(y - A) + B\,y$$

$$1 = -CA$$

$$C = -\frac{1}{A}$$

$$B + C = 0 \quad B = -C = \frac{1}{A}$$

(4)

$$\int_1^{e^{-kx}} \left(\frac{1}{y - A} - \frac{1}{y}\right) dy = -A\,k(1 - a)\frac{I}{i}\,t \quad \text{or}$$

[1] *See* p. 11 of Reprint from *J.S.C.I.*, 31st May, 1890.

$$\log. \frac{e^{-kx} - e^{-ka}}{e^{-kx}(1 - e^{-ka})} = -Ak(1-a)\frac{1}{i}t$$

$$\log. \frac{e^{-kx}(1 - e^{-ka})}{e^{-kx} - e^{-ka}} = e^{-ka}k(1-a)\frac{1}{i}t$$

Multiply by e^{ka} gives—

$$\log. \frac{1 - e^{-ka}}{1 - e^{-ka}e^{kx}} \text{ and by } e^{ka}$$

$$\log. \frac{e^{ka} - 1}{e^{ka} - e^{-kx}} = \frac{k}{e^{ka}}(1-a)\frac{1}{i}t$$

Substituting O for e^{ka} and D for kx, we have—

$$\frac{O - 1}{O - \epsilon^{D}} = \epsilon^{\frac{k}{O}(1-a)\frac{1}{i}t}$$

collecting symbols $k\ (1 - a)$ and i into i_o—[1]

$$\frac{O - 1}{O - \epsilon^{D}} = \epsilon^{\frac{1}{O}\frac{1}{i}t} \quad \text{or}$$

$$O - \epsilon^{D} = (O - 1)\,\epsilon^{-\frac{1}{O}\frac{I}{i}t}$$

$$\epsilon^{D} = O - (O - 1)\,\epsilon^{-\frac{1}{o}\frac{I}{i}t}$$

$$D = \log.\left[O - (O - 1)\,\epsilon^{-\frac{1}{o}\frac{I}{i}t}\right].$$

The Point of Inflexion on the Curve.[2]

Dr. Hurter's Note Book, H.N.—C., pp. 69–75.

$$D = \log.\left[O - (O - 1)\,\epsilon^{-\frac{It}{io}}\right].$$

Put $\frac{It}{io} = a$, then—

$$D = \log.\,[O\,(1 - \epsilon^{-a}) + \epsilon^{-a}].$$

Differentiate according to log. a as follows :—

If $u = O\,(1 - \epsilon^{-a}) + \epsilon^{-a}$

then $D = \log._{\epsilon} u$ and

$$\frac{dD}{du} = \frac{1}{u} = \frac{1}{O\,(1 - \epsilon^{-a}) + \epsilon^{-a}},$$

[1] The MS. reads i_o, but the sense seems to require io.

[2] Compare Sheppard and Mees, *loc. cit.*

$$\frac{du}{da} = O\epsilon^{-a} - \epsilon^{-a} \text{ hence } \frac{dD}{da} = \frac{O\epsilon^{-a} - \epsilon^{-a}}{O(1 - \epsilon^{-a}) + \epsilon^{-a}},$$

$$\frac{dD}{da} = \frac{O\epsilon^{-a} - \epsilon^{-a}}{O - (O\epsilon^{-a} - \epsilon^{-a})}.$$

If $m = \log._\epsilon a$, then $a = \epsilon^m$; hence—

$$\frac{da}{dm} = \epsilon^m = a \text{; hence}$$

$$\frac{dD}{dm} = \frac{O\epsilon^{-a} - \epsilon^{-a}}{O - (O\epsilon^{-a} - \epsilon^{-a})} .a$$

this is the tangent of the curve when log. a is made abscissa and D ordinate.

This tangent is important.

For if a is 0, the tangent is 0, and when $a = \infty$ the tangent is 0 again, for the fraction may be written—

$$tg = \frac{\epsilon^{-a}(O - 1)}{O - \epsilon^{-a}(O - 1)} a \text{ or}$$

$$tg = \frac{O - 1}{O\,\epsilon^{a} - (O - 1)} a.$$

Since a can only have positive values, and $O\,\epsilon^a$ is always greater than $O - 1$, the values are always positive, and vary from O to a maximum, decreasing again to O. This maximum will be obtained when $\dfrac{a}{O\,\epsilon^a - (O - 1)}$ is itself a maximum, or when $\dfrac{d}{da}\dfrac{a}{O\,\epsilon^a - (O - 1)} = 0$, but this is the case when

$$(1 - a)\cdot\epsilon^a = \frac{O - 1}{O}.$$

This is the point of double curvature.

The density at the point of double flexure is—

$$D = \log.\left[1 - \left(\frac{O - 1}{O}\right)\epsilon^{-a}\right] + \log. O.$$

but $\dfrac{O - 1}{O} = \epsilon^a\ (1 - a)$, consequently—

$$D = \log.\left[1 - (1 - a)\right] + \log. O$$

$$D = \log. a + \log. O.$$

The tangent of the angle at that point is—

$$tg = \gamma = \frac{\epsilon^{-a}}{\left(\dfrac{O}{O - 1}\right) - \epsilon^{-a}} .a.$$

Substituting for $\frac{O}{O-1}$ its equal value $\epsilon^{a}(1-a)$ we get—

$$\gamma = \frac{\epsilon^{-a}\epsilon^{a}(1-a)}{1-\epsilon^{-a}\epsilon^{a}(1-a)}a$$

$$= \frac{1-a}{1-(1-a)}a = 1-a.$$

These two values are important, viz. :—

$$\left.\begin{array}{l} D = \log.(aO) \\ tg = \gamma = 1-a \end{array}\right\} \text{for point of double flexure.}$$

(*Copy.*)

WEST SHANDON HOUSE,
SHANDON, N.B.
22nd August, 1897.

MY DEAR DRIFFIELD,—

Here we have been in a lovely spot for now five days, and there has not been a day that heaven did not weep over "the time wasted." It pours every day, if not all day, yet at times so hard that being out is unpleasant. Still, we have made some nice excursions, and if ever you want to do a week's camera work on some worthy landscapes this is the place for you to come to. There are plenty of cameras at work, but most of them without any brains or knowledge to work them. I have kept my word and thought of nothing. But to-day being Sunday, and having been to church and listened to a grand long sermon on "being nigh to the Kingdom of God," I could not help my thoughts wandering, and here is what half an hour's quiet produced on paper. As I have no experimental evidence here I cannot go on, but I will just impart to you the thought for your criticism and consideration. The chaos of experimental evidence which you have accumulated wants an analytical expression to sift what is due to development of latent image from that due to fog, and to bring clearly out what is simply due to alteration of developer. You know already that the *starvation* theory did not satisfactorily explain matters, and that diffusion seems to work so fast, that there is no need for assuming a limit to the development for want of developer. Equally (in the opposite direction) there is not clear evidence that the *bromide* generated in the film by decomposition of bromide of silver materially affects results, since it escapes by diffusion just as fast as the pyro can enter.

Still, to some extent, all these phenomena will affect the results, and these results will naturally deviate in consequence from any simple rule, which cannot possibly be so formulated as to take account of everything—constitution of film, quality of gelatine, velocity of diffusion to and fro, temperature, etc.

All that mathematical analysis can do is to trace the *main* features of the course of development.

The following thoughts have thus taken shape.

Let the total quantity of silver on any part of the plate be A. After illumination this consists of two portions, unaffected bromide = B and an affected part a, and always will, $B + a = A$. *The difference between the two portions is not one of kind, but simply one of degree in rapidity of* development,[1] and this thought applied to the density of the resulting image makes that (in imagination) also consist of two parts, *i.e.*, of developed *fog* and developed *image*. Call the one x (image) and y the fog, then the density will be $x + y = D$.

The small amount of latent image developed at any moment will be, say—dx and that amount is developed in the moment dt, hence the velocity of development is expressed as $\frac{dx}{dt}\ \frac{\text{(image)}}{\text{(time)}}$.

This velocity will be proportional to a factor k_1, which depends upon the constitution of the developer, and to the amount of undeveloped latent image present at that moment which we are considering. Let the latent image already developed be x, then $(a - x)$ is the amount of undeveloped latent image and the velocity of development (growth of density in unit time) is—

$$\frac{dx}{dt} = k_1 (a - x).$$

But while the latent image is thus developing, the ordinary bromide of silver B is attacked as well, and at the moment we are considering the density of fog is already y, so that $B - y$ is the amount of unaltered bromide not yet developed. The rapidity of development will depend upon a factor k_2, depending upon the developer, and also, like the former, proportional to the bromide still present.

We have thus for fog-velocity—

$$\frac{dy}{dt} = k_2 (B - y).$$ (Similar in kind, different in degree from the other!)

By integration, this gives for the latent image density in time t—

$$x = a\,(1 - \epsilon^{-k_1 t})$$

and for fog—

$$y = B\,(1 - \epsilon^{-k_2 t})$$

and the total developed density at time t is the sum of these two.

This sum expressed in its simplest form comes out—

$$D = A\left(1 - \epsilon^{-k_2 t}\right) + a\,(\epsilon^{-k_2 t} - \epsilon^{-k_1 t})$$

or

$$D = A\left(1 - \epsilon^{-k_2 t}\right) + a\left(\frac{\epsilon^{k_1} - \epsilon^{k_2}}{\epsilon^{k_1 + k_2}}\right)^t$$

[1] *See pp.* 88, 227, 231.

This shows how, with a given developer, the whole density grows. The first term is the fog, and is a constant and an excuse for deducting it as we have done, but the second term is also affected by that fog, hence this deduction is not permissible when the fog is at all appreciable. If a developer be *so* constituted that within the time of development there is no fog, that means that k_2 is $= 0$, and the above expression changes to—

$$D = a\,(1 - \epsilon^{-k_1 t})$$

and if ϵ^{-k_1} is put as in our original paper $= \alpha$, then we have $D = a\,(1 - \alpha^t,)$ which is the formula *exactly* of our original paper (a) being the highest possible density reachable for the particular exposure with the particular developer (a non-fogging one !).

There now remains only to replace k_1 and k_2 by functions which depend upon the composition of developers. This I do not think is difficult when I get home. I cannot do it here. You may see that now I shall be able to put order into the whole thing. I expect that—

$$k_1 = \frac{\text{Pyro} \times \text{Alkali}}{\text{Bromide}} \times \text{constant}$$

$$k_2 = \text{the same} \times \text{other constant},$$

and that the two constants are given by the speed of plate, thus—

$$\frac{\text{Constant No. 1}}{\text{Constant No. 2}} = \text{speed of plate},$$

or it may be that Constant No. 1 alone will be affected by the speed and Constant No. 2 remain. That I can only tell from the experiments.

But I think I shall now be able to put matters straight. Please keep this letter ; it is the only record in ink of this thought, and I may lose my pencil copy.

Trusting all is going on well, and that you are not working unnecessarily at this subject.

I remain,

Yours sincerely,

F. HURTER.

Give Mrs. Hurter's, Annie's, and James's love to May. They all wish to be kindly remembered by you.—F. H.

One very important experiment is to be made.

Unexposed strips of Cadett 21 and Cadett 103 or 111 are to be developed for 2, 4, 8 minutes in a fogging developer but together ! Can you do that ?

(To find fog related to speed—if it is.)—F.H.

1

$$\boxed{dx = \frac{I}{i}(I-a)\left[e^{-kx} - e^{-ka}\right] dt}$$ (p.35)

Dividing both sides by $(e^{-kx} - e^{-ka})$

$$\Rightarrow \frac{dx}{(e^{-kx} - e^{-ka})} = \frac{I}{i}(I-a)\, dt$$

Taking the integral of both sides

$$\Rightarrow \boxed{\int_0^x \frac{dx}{e^{-kx} - e^{-ka}} = \int_0^t \frac{I}{i}(I-a)\, dt} \quad (1)$$

Let $y = e^{-kx}$ and $A = e^{-ka}$

$$\Rightarrow dy = -k e^{-kx}\, dx$$

$$\Rightarrow dx = -\frac{dy}{k e^{-kx}}$$

$$\Rightarrow dx = -\frac{dy}{ky} \qquad \text{since } y = e^{-kx}$$

check limits of $\int_0^x$

When $x = 0$

$\Rightarrow y = e^{-k(0)}$

$\Rightarrow y = e^0$

$\Rightarrow y = 1$

When $x = x$

$\Rightarrow y = e^{-kx}$

Hence, equation (1) becomes

$$\int_1^{e^{-kx}} -\frac{dy}{ky}\cdot\frac{1}{(y-A)} = \int_0^t \frac{I}{i}(I-a)\, dt$$

2

$$\Rightarrow \int_1^{e^{-kx}} \frac{dy}{y(y-A)} = -k\frac{I}{i}(I-a)\int_0^t dt$$

$$\Rightarrow \int_1^{e^{-kx}} \frac{dy}{y(y-A)} = -k\frac{I}{i}(I-a)\left[t-0\right]$$

$$\Rightarrow \boxed{\int_1^{e^{-kx}} \frac{dy}{y(y-A)} = -k(I-a)\frac{It}{i}} \qquad (2)$$

Solving the LHS using partial fractions

Let $\frac{1}{y(y-A)} = \frac{C}{y} + \frac{B}{y-A}$ (i)

Multiplying each term by $y(y-A)$

$$1 = C(y-A) + By \qquad \text{(ii)}$$

Now find the values of C and B:

C: Let y be $\emptyset$ in (ii)

$$\Rightarrow 1 = C(\emptyset - A) + B(\emptyset)$$

$$\Rightarrow 1 = -CA$$

$$\Rightarrow C = -\frac{1}{A}$$

B: Let y be A in (ii)

$$\Rightarrow 1 = C(A-A) + BA$$

$$\Rightarrow 1 = BA$$

$$\Rightarrow B = \frac{1}{A}$$

3

Put values for C and B into (i)

$$\frac{1}{y(y-A)} = -\frac{1}{Ay} + \frac{1}{A(y-A)}$$

$$\Rightarrow \frac{1}{y(y-A)} = \frac{1}{A}\left(\frac{1}{y-A} - \frac{1}{y}\right)$$

Hence the integral in (2) takes the form

$$\int_1^{e^{-kx}} \frac{1}{A}\left(\frac{1}{y-A} - \frac{1}{y}\right) dy = -k(I-a)\frac{It}{i}$$

Shifting A to the RHS:

$$\int_1^{e^{-kx}} \left(\frac{1}{y-A} - \frac{1}{y}\right) dy = -kA(I-a)\frac{It}{i} \quad (3)$$

Taking the LHS of (3):

$$\int_1^{e^{-kx}} \left(\frac{1}{y-A} - \frac{1}{y}\right) dy$$

$$\Rightarrow \int_1^{e^{-kx}} \frac{1}{y-A}\, dy - \int_1^{e^{-kx}} \frac{1}{y}\, dy$$

$$\Rightarrow \log(y-A)\Big|_1^{e^{-kx}} - \log y\,\Big|_1^{e^{-kx}}$$

Since:

$$\int \frac{1}{x-a}\,dx = \log(x-a)$$

$$\int \frac{1}{x}\,dx = \log x$$

4

Putting in the limits:

$$\Rightarrow \left[\log(e^{-kx} - A) - \log(1 - A)\right] - \left[\log(e^{-kx}) - \log(1)\right]$$

$$\Rightarrow \log\left(\frac{e^{-kx} - A}{1 - A}\right) - \log\left(\frac{e^{-kx}}{1}\right)$$

$$\Rightarrow \log\left(\frac{e^{-kx} - A}{e^{-kx}(1 - A)}\right)$$

Since:
$\log x - \log y - \log z$
$\Rightarrow \log\left(\frac{x}{yz}\right)$

Hence eqn. 3 takes the form:

$$\boxed{\log\left(\frac{e^{-kx} - A}{e^{-kx}(1 - A)}\right) = -kA(I - a)\frac{It}{i}}$$

Putting $A = e^{-ka}$ into this formula

$$\log\left(\frac{e^{-kx} - e^{-ka}}{e^{-kx}(1 - e^{-ka})}\right) = -ke^{-ka}(I - a)\frac{It}{i}$$

Using log rule below:

$$-\log\left(\frac{e^{-kx}(1 - e^{-ka})}{e^{-kx} - e^{-ka}}\right) = -ke^{-ka}(I - a)\frac{It}{i}$$

$$\log\left(\frac{e^{-kx}(1 - e^{-ka})}{e^{-kx} - e^{-ka}}\right) = ke^{-ka}(I - a)\frac{It}{i}$$

Since
$\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)$
where $a, b > 0$

Mult LHS by e^{kx}:

$$\Rightarrow \log\left(\frac{e^{kx}e^{-kx}(1-e^{-ka})}{e^{kx}e^{-kx} - e^{kx}e^{-ka}}\right) = ke^{-ka}(I-a)\frac{It}{i}$$

$$\Rightarrow \log\left(\frac{1-e^{-ka}}{1-e^{kx}e^{-ka}}\right) = ke^{-ka}(I-a)\frac{It}{i}$$

Mult by e^{ka}:

$$\Rightarrow \log\left(\frac{e^{ka}(1-e^{-ka})}{e^{ka}(1-e^{kx}e^{-ka})}\right) = ke^{-ka}(I-a)\frac{It}{i}$$

$$\Rightarrow \boxed{\log\left(\frac{e^{ka}-1}{e^{ka}-e^{kx}}\right) = \frac{k}{e^{ka}}(I-a)\frac{It}{i}}$$ p.36

<u>NOTE</u> Thanks to Dr Khalid Khan of Malakand, Pakistan
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